The Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx,$$

So today, we're not going to start by reading out a formula, because the Fourier transform uses calculus – and really calculus is applied geometry, where you're either deriving the gradient of a curve or you're integrating the area underneath that curve. So the terms used in an equation don't make a lot of sense when they are read out, you need to show how the equation works graphically – that is, on a graph. Specifically, on a graph that has an x axis that runs horizontally and a y axis that runs vertically – because upon such a graph you can display a function.

A function is the relation between a set of inputs and a set of outputs. So if a function is to double everything and your inputs are 1, 2, 3 and 4, then your outputs will be 2, 4, 6 and 8. So, mathematically that function can be expressed as y (the output) equals 2 times x (the input) and if you plot a line y = 2x on a graph with an x and a y axis you'll get a straight line with a gradient of 2.

Of course, in the real world, or at least in the natural world, straight lines aren't very common – but functions are very common. Lots of thing things that happen in the real world can be portrayed on graphs with x and y axes – where the x axis represents time and the y axis represents the magnitude of whatever it is you are measuring over time. So, you could chart 74 minutes of Arecibo radio telescope tracking data or you could chart 74 minutes of Beethoven's ninth symphony. In either case you get a waveform with different amplitudes and varying frequencies – which just means particular amplitude peaks will sometimes appear more frequently and sometimes less frequently. But, whether you are measuring electromagnetic radiation or measuring the oscillations in atmospheric pressure that create sound – you'll get a waveform that is, in essence, a function.

And this is where the Fourier Transform comes into play. Faced with 74 minutes of Beethoven, that might only make sense to a human, or 74 minutes of Arecibo radio telescope data that might only make sense to an alien – we can apply the universal translator known as mathematics. The Fourier transform can convert a function of magnitude over time into a function of magnitude over frequency. So rather than seeing the 74 minutes of Beethoven's audio displayed on a very long graph, you would have a shorter graph that just shows the frequency distribution of the whole piece. So, rather than listening through the whole 74 minutes, you can analyse its component frequencies at a glance, identifying the dominant frequencies as well as some minor irregular frequencies that are probably just background noise. The same transform of the Arecibo telescope data might allow you to filter out the frequencies of known natural sources, like pulsars and quasars, and then search through what's left for signs of extra-terrestrial intelligence.

The Fourier transform is more of a mathematical tool than it is a physics formula, but its application is so entrenched in physics that it deserves special mention in Fantastic Physics Formulas anyway. And in keeping with the theme of the show, here is the formula – though as we say, it makes more sense when portrayed on a graph than it does in audio.

The Transformed function x-prime, representing the magnitude and phase of a chosen frequency, is equal to area under the curve of the original function x, times e to the power of 2 times Pi times i times x times the chosen frequency. This is all where x represents the interval you are sampling and i is an imaginary number – essentially the square root of minus one.

And, after you've run the equation once, you have to run it again and again, for all the other major frequencies in your original function.

So, as well as the Fourier transform being graphical, it's also an algorithm – that is, a systematic way of approximating something. This means it's not a formula where you just plug in some numbers and get an answer. You first have to decide on a dominant frequency to sample and then pick a second frequency to sample and so on – and if you are analysing 74 minutes of data you may have to run the formula across several different time intervals as well. So, people often build computer programs that will do their Fourier number-crunching for them – but the first step is always a human-level decision about which parameters to set and the last step is always a human consideration of whether the algorithmic approximation you ended up with approximates what you were looking for. So, this is a formula that melds together both human judgement and mathematics – and that is kind-of fantastic.

The Hawking radiation formula

$$T = \frac{\hbar c^3}{8\pi G M k_B} \quad \left(\approx \frac{1.227 \times 10^{23}\ \mathrm{kg}}{M}\ \mathrm{K} = 6.169 \times 10^{-8}\ \mathrm{K} \times \frac{M_\odot}{M} \right),$$

We generally think of black holes as bottomless gravity wells that suck things down, never to be seen again, But, Professor Stephen Hawking has argued that there should be a certain radiative loss from any black hole and he went on to develop a formula that expressed that radiative loss as a temperature, where that temperature T equals h-bar, which is what's known as Planck's reduced constant, multiplied by c cubed (yep, that's the speed of light cubed),. And all that is divided by M the mass of the black hole, times 8 times pi times G times Kb – where Pi is pi, G is the gravitational constant and Kb is Boltzmann's constant.

As much as this humble podcast can explain such complex math, the references to Planck's reduced constant indicates quantized energy is central to the calculation and Boltzmann's constant enables that quantized energy to be converted into a temperature. And since we are dealing with the energy output of a spherical object, the surface area of that spherical object is key to the effect – which is why pi is there. And for any very dense object like a black hole, there is a direct relationship between its mass and its surface area – which is why M for mass and G the gravitational constant are also there.

But remember that the formula is actually measuring a temperature. It turns out that when you run the formula many of the units of the terms in the equation cancel each other out. So where you have Plank's reduced constant, hbar, times c cubed - these are expressed in units of energy and distance and time. And in the formula, hbar times c cubed is divided by G and Boltzmann's constant, which also have units of energy and distance and time. So, in dividing the one set of terms by the other set of terms – most of measurement units of those terms just cancel each other out. Indeed once you work through all that math and all the various units that do cancel each other out , you are just left with the temperature component of Boltzmann's constant, expressed in Kelvin, and its relationship to the mass of the black hole you are calculating the temperature of

And this is what it's all about – the amount of Hawking radiation emanating from a black hole, which is measured as the temperature of that black hole, is ultimately determined by the black hole's mass, although its actually an inverse relationship – so the bigger the black hole is, the less Hawking radition it radiates – that is, the cooler it is. A black hole with the mass of the Sun, would have a temperature of 6.169 times 10 to the minus 8 Kelvin, which is a temperature well less than 1 Kelvin. And since all the black holes we know about have masses greater than the Sun, they will all have temperatures that are way less than 1 Kelvin.

The proposed mechanism that underlies Hawking radiation involves a unique set of circumstances that could only happen directly adjacent to a black hole's event horizon. The classical understanding of an event horizon is that once something goes in, it will never come out again. But, Hawking proposed that quantum-level phenomena are universal, including the phenomenon where paired particles and anti-particles can suddenly appear out of nowhere and then disappear again in a puff of mutual annihilation so if such events are universal they will also happen directly-adjacent to a black hole event horizon. In that case, one half of that pair might get sucked in while the other remains outside. So rather than a random and self-cancelling event taking place, the Universe gains a new particle – while the sucked-in antiparticle annihilates within the event horizon causing a reduction in the black hole's mass.

At least that is one physical interpretation of the complex relationships that are captured in the Hawking radiation formula. Another interpretation involves a particle from the black hole quantum-tunneling its way out through the event horizon. A third interpretation involves the black hole repaying a negative energy debt to the wider Universe – whatever the heck that means.

Nonetheless, whatever physical processes may underlie it, the formula itself has persisted in the face of an extraordinary level of very-public peer review. The ground-breaking nature of the Hawking radiation formula is that it establishes a link between the sub-atomic strangeness of quantum mechanics and some very real and astronomically-sized objects that are generally only characterized by using the space-time curvature of relativity physics. That melding of some otherwise-unmeldable physics theory is more than enough reason to call this formula just a little bit fantastic.