Free-fall time to a centre of mass

$$\frac{\pi}{2\cdot\sqrt{2}}\cdot\sqrt{\frac{D^3}{G\cdot M}}$$

A listener asked the question “How long would it take an object to fall into the Sun from one light year away”. The answer to this question turns out to be a Fantastic Physics Formula, which has been attributed to Arthur C Clarke (who also had geostationary orbits attributed to him). In reality, it could just as well be attributed to Johannes Kepler, since it’s really about the mathematics of an orbital system and the relationship between the distance of an object from the centre of mass of an orbital system and the orbital period of that object.

When we talk about something falling in towards something else, that’s still an orbital system. The Moon is falling in towards the Earth, it’s just falling at such an angle that it keeps missing and comes around again for another try. So, if you do want something to fall straight in to a centre of mass you need to adjust its orbit so it will do exactly that.

Kepler’s Third Law of planetary motion, says the square of the orbital period of a planet around its star is directly proportional to the cube of the semi-major axis of its orbit. The semi major axis of a circular orbit is its radius, while the semi-major axis of an elliptical orbit is half of the longest axis of its orbit.

The actual formula for Kepler’s Third law is T (the period of the orbit) squared, equals 4 times pi squared times the semi-major axis cubed divided by G, the gravitational constant, divided by the sum of the two masses in the orbital system.

You can convert that formula into a formula that calculates the orbital period of an orbit that’s a straight line between the falling object and the centre of mass of an orbital system. That formula is T (the period of the fall) equals pi divided by 2 times the square root of two, all times the square root of the semi-major axis squared divided by G, the gravitational constant, times the sum of the two masses in the orbital system. The main change to the standard form of Kepler’s Third Law is that you’re only going to measure the object’s straight-line fall in, but not its straight-line return back that would complete a full orbit.

Anyway using this variant form of Kepler’s Third Law, if the two masses are the Sun plus the relatively inconsequential mass of the falling object then the period of the fall from one light year out works out to be 2.8 million years.

We have to make a few assumptions to make this calculation work. Firstly, it is a two body problem and we are assuming nothing else will get in the way of the object that falls a whole light year into the Sun. This is entirely possible since space is so vast, but it’s not guaranteed. What’s less plausible is the situation in which an object has no independent motion of its own relative to the Sun and just falls straight in. In reality, any objects falling into the Sun’s gravity well will more likely spiral in and hence have much more prolonged falls. Lastly, the falling object won’t actually make it to the centre of the Sun, it will slowed by the corona and disintegrate well before that, but that’s a tiny proportion of its light year long fall.

This free fall time formula that derives from the formula of Kepler’s Third Law has broader uses. When you boil down the maths, it works out that the period of a straight line fall from a set distance is always around 18 per cent of what the orbital period of a circular orbit at that same distance would be. So, for example, given the orbital period of the Earth is one year – and its orbit is a *roughly* circular orbit, then the straight-line free fall time from the Earth's orbit into the Sun would be around 65 days, because 65 days is around 18 per cent of one year.

The formula also has a number of applications in cosmology. For example, it can be used to estimate the time required for a dust cloud to collapse in on itself, perhaps to form a new star. It can also estimate the time required between the commencement of the core collapse of a massive star and its subsequent destruction in a supernova explosion. In both cases, the math only works if you assume the absence of any countering outward forces, but as a general approximation, it's not half bad.

So in this episode the title of Fantastic Physics Formula should really go to Kepler's Third law of planetary motion, but we hope we've demonstrated here how you can take one formula and reconfigure it a bit so that it does quite a different job and does it well. Even if that's not fantastic, it is kind of useful.

The Principle of Least Action

$$\text{Action} = S = \int_{t_1}^{t_2} (\text{KE} - \text{PE})\, dt.$$

The principle of least action defines the path that an object will follow in time and space. It can also be used for other things, like defining the shape that something flexible will adopt in a gravity field under particular conditions, but defining the path of a projectile is its most common usage

If you imagine throwing a ball – and you do it in a uniform gravity field and assume little or no wind – then the path it follows will be a parabola. This is something that Newton's second Law of Motion, f = ma could tell you just as well. But, the Principle of Least Action is all about the action while Newton is all about the forces. In simple scenarios like throwing a ball, the distinction doesn't really matter, but when you want to describe more complex scenario, like a pendulum attached to the end of another pendulum apparently least action math is the way to go.

But let's stick with throwing a ball – if you throw it hard it's going to go higher, because you've given it more kinetic energy and the path it follows maximizes its potential energy, since in gaining height it's gaining gravitational potential energy.

So while you could use F=ma to calculate the acceleration of the ball at each point in time and hence define its path, a parabola you can also calculate the ball's Action as being equal to its Kinetic Energy minus its Potential Energy – and do that at each point in time from the start to the end of its path. Then equation to calculate that path is S (action) equals the integral from t1 to t2 of Kinetic Energy minus Potential Energy times dt – meaning its being integrated over time, t. Indeed, it's specifically being integrated over the time period between t1 and t2. We use integration because it's a continuous process and we're integrating it over time because for each point in time (between t1 and t2) there's an overarching principle or an effect unfolding which is the Principle of Least Action.

Not only are we saying that at each and every point in time the position of the thrown ball is determined by Kinetic Energy minus Potential Energy, but there's also an overarching principle where the calculated action at each and every point will always be the *least possible* action. So, in the example of throwing the ball, the calculated action at each point in time along the path of the object will the least value that it possibly can be – and the result will be a parabolic path. If you throw it hard and high, or low and soft, the principle will still apply and you'll still get a parabolic path, but one will have a higher altitude than the other. So, the initial conditions working on the thrown ball still matter but the Principle of Least Action will apply from the moment it leaves your hand. As Principle of Least Action proponents like to say: the Universe is lazy – it will only ever do as much as it absolutely has to do.

The understanding that for all possible paths between t1 and t2, there is only one true and possible path, the path of least action, the lazy way, is a significant fundamental concept of mechanics. You can apply the principle to the action of any mechanical system and derive the equations of motion for that system. The equations of motion for a system are whatever equations define the system's motion over time. These can be straightforward functions drawn on x and y axes where x is distance over y time – so for a particle moving at a constant velocity in a vacuum you get a straight line. But if there's dynamic motion involved, like when a thrown ball follows a parabolic curve in a gravity field, then the equations of motion are differential equations. And you can take a step further and swap Euclidean space for curved spacetime and derive relativistic equations of motion.

It's probably a step to far to say the principle of least action underlies everything that matters in mechanics theory. It's better thought of as a useful perspective, which may be particularly useful where formulas that calculate what's happening at a single point in time (like f=ma) aren't proving useful. The whole idea of least action dates back to the 1700s and is generally attributed to Maupertuis, but with due mention owed to Hamilton, Lagrange, Euler and maybe Leibniz too. So, it's hardly a new idea – it's more what you'd call an oldie but a goodie.